Groebner Bases for Real Flag Manifold $F(1,\,1,\,1,\,m-3)$

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Abstract
The $mod\ 2$ cohomology of the real flag manifold $F(n_1, n_2, \cdots, n_k)$ is known to be isomorphic to a polynomial algebra modulo certain ideal. In this paper, the Groebner bases for these ideals are obtained in the case of the real flag manifold $F(1,1,1,\,m-3)$. We further apply the reduced Groebner bases to compute the height of the cohomology classes and determine which of the classes vanish.

Keywords and phrases: Groebner basis, cohomology, flag manifolds, dual Stiefel-Whitney classes

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Introduction

Given a sequence of positive integers $n_1, n_2, \cdots, n_k$ the real flag manifold $F(n_1, n_2, \cdots, n_k)$, $k \geq 2$ is defined as the set of flags of type $(n_1, n_2, \cdots, n_k)$ in R^m where $m = (n_1, n_2, \cdots, n_k)$ is a k-tuple $(V_1, V_2, \ldots, V_k)$ of mutually orthogonal subspaces in R^m such that dim $(V_1) = n_i$.

The real flag manifold $F(n_1, n_2, \cdots, n_k)$ consists of all flags of type ($n_1, n_2, \cdots, n_k)$ which can be naturally identified with the homogeneous space

$$\frac{O(m)}{O(n_1) \times O(n_2) \times \cdots O(n_k)}.$$

This is a generalization of the Grassmann manifold $F(n_1, n_2)$ of n_1 planes in R^m. When $n_1 = n_2 = \cdots = n_k = 1$, the corresponding manifold is called the complete flag manifold of length m. Flag manifolds are important and arise naturally in many areas of study like differential geometry, representation theory and algebraic geometry. The Grassmann manifold are central objects in geometry and topology. By Borel's description (Borel (1953), the mod-2 cohomology algebra of $F(n_1, n_2, \cdots, n_k)$ is the

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polynomial algebra in Stiefel-Whitney classes of canonical vector bundles γ_1, γ_2, ... , γ_{k+1} over $F(n_1,\ n_2,\ \cdots,\ n_k)$ modulo the ideal $I_{n_1,n_2,\cdots,n_k}$ generated by the dual Stiefel-Whitney classes.

However, this description does not provide an efficient algorithm for deciding whether a certain polynomial in these Stiefel–Whitney classes is the zero cohomology class or not. The determination of non-vanishing Stiefel-Whitney classes is very important in the study of vector fields, non-immersion and non-embeddings in R^m and the cohomology height of manifolds.

Interestingly, the method of Groebner bases is applicable in calculating the cohomology of manifolds as a quotient of a polynomial algebra. Petrovic and Prvulovic (2011) and Petrovic, Prvulovic, Radovanovic, (2013) determined the cohomology of various Grassmann manifolds using Groebner basis as tools while Shimkus (2010) used Groebner bases to obtain a simple description of the cohomology of incomplete flag manifold of length 2, $H^*(F(1,1,\ n-2),\ Z_2)$ for certain n and their nonimmersion results.

The combinatorics tools of Groebner bases will be adopted to compute the ideals $I_{n_1,n_2,\cdots,n_k}$ generated by the dual Stiefel-Whitney classes for $n_1 = n_2 = \cdots = n_k = 1$. The main results are the exhibition of the reduced Groebner basis for $I_{n_1,n_2,\cdots,n_k}$ and its application on the cohomology height of the manifold.

In section 2 we give some preliminary purely algebraic results on Groebner basis and cohomology of flag manifolds. The section will begin with some basic concepts from the theory of Groebner bases and delve extensively into relevant ideas and point out a few elementary facts that will be used in proving our results. The reduced Groebner bases for ideals $I_{n_1,n_2,\cdots,n_k}$ will be constructed in section 3, for all $m \geq 4$ and an additive bases for the cohomology algebra $H^*(F(1,1,1,\ n-3),\ Z_2)$ will be given. We used these Groebner bases to determine the height of the cohomology classes.

Preliminaries and Definitions

The incomplete real flag manifold of length r,

$$F\,(\underbrace{1,1,\ \dots\ ,1,}_{r-times}\ m-r\ ) \cong \frac{O(m)}{\underbrace{O(1)\times\dots\times O(1)}_{r-times}\times O(m-r)}$$

is a smooth, connected, compact homogeneous manifold of dimension $\frac{2nr-r(r+1)}{2}$ (Ajayi (2001)). Thus $F(1,1,1, n-3)$ is of dimension $3(n-2)$. Let k be a field and $k[x_1, x_2, \cdots, x_n]$ be the polynomial algebra in n indeterminates. A monomial in the variables $x_1, x_2, \cdots, x_n$ is a power product $x_1^{a_1}, \ldots, x_n^{a_n}$ where $a_i \geq 0$ for $i = 1, \ldots, n$ The set of all monomials is denoted by M. A term in $k[x_1, x_2, \cdots, x_n]$ is a product of $\beta \in k$ and $m \in M$. Thus a polynomial $f \in k[x_1, x_2, \cdots, x_n]$ is defined as

$$f = \sum_{i=1}^{r} \beta_i m_i$$

where m_i are pairwise different monomials and $\beta_i \in k\{0\}$. The real flag manifold $F(1,1,1, m-3)$, $m \geq 4$ is a manifold defined as the set of flags of type $(1,1,1, m-3)$ of mutually orthogonal subspaces in R^n .

The manifold structure is obtained from the natural identification

$$F(1,1,1, m-3) \cong \frac{O(m)}{O(1) \times O(1) \times O(1) \times O(m-3)}.$$

Over this manifold are subspaces(l_1, l_2, l_3, V), where l_i ‘s are mutually orthogonal lines through the origin in R^m, and V is the (m − 3)-dimensional subspaces of R^m, orthogonal to $l_1, l_2, and\ l_3$.

Results

The aim here is to compute a Groebner basis for the ideal $I_{1,1,1,m-3}$. All computation will be in modulo 2. Given any $\alpha, \beta \in Z$ the binomial coefficient is defined by

$$\left(\frac{\alpha}{\beta}\right) = \begin{cases} \frac{\alpha(\alpha-1)\ldots(\alpha-\beta+1)}{\beta!} & \beta > 0 \quad 1 \qquad \beta \\ = 0 \quad 0 & \beta < 0 \end{cases}$$

(see Radovanovic (2016)). The formula

$$\left(\frac{\alpha}{\beta}\right) = \left(\frac{\alpha-1}{\beta}\right) + \left(\frac{\alpha-1}{\beta-1}\right)$$

is valid for all $\alpha, \beta \in Z$. Thus,

$$\left(\frac{\alpha-1}{\beta-1}\right) \equiv \left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha-1}{\beta}\right)\ mod\ 2$$

For $F(1,1,1, m-3), m > 3$, it is well known that the $(mod\ 2)$ cohomology algebra is isomorphic to the quotient algebra $Z_2[x_1, x_2, x_3]/I_{1,1,1,m-3}$ (Ajayi, 2001). Also, by Borel's description (Borel, 1953) the $mod\ 2$ cohomology algebra of $F(1,1,1, m-3)$ is the polynomial algebra on the Stiefel-Whitney classes, where $x_1, x_2, x_3 \in H^1F(1,1,1, m-3)$ are Stiefel-Whitney classes of three canonical line bundles $\gamma_1, \gamma_2, \gamma_3$ over $F(1,1,1, m-3)$, that is, $x_1 = \omega_1(\gamma_1),\ x_2 = \omega_1(\gamma_2),\ x_3 = \omega_1(\gamma_3)$. Thus, $I_{1,1,1,m-3} = (z_{m+1}, z_{m+2}, z_{m+3})$ is the ideal in $Z_2[x_1, x_2, x_3]$, generated by the dual classes $z_{m+1}, z_{m+2}, z_{m+3}$.

These dual classes are the dual to the Stiefel-Whitney classes of the Whitney sum of the three canonical line bundles, which are obtained from the equation

$$(1 + x_1 + x_2 + x_3)(1 + \underline{x_1} + \underline{x_2} + \underline{x_3}) = 1$$

or

$$(1 + \underline{x_1} + \underline{x_2} + \underline{x_3}) = (1 + x_1 + x_2 + x_3)^{-1}$$

that is,

$$1 + \underline{x_1} + \underline{x_2} + \underline{x_3} + \cdots = (1 + x_1)^{-1}(1 + x_2)^{-1}(1 + x_3)^{-1}$$

Putting,

$$\underline{x_1} = z_1, \qquad \underline{x_2} = z_2, \qquad \underline{x_3} = z_3$$

Then,

$$\begin{aligned}1 + z_1 + z_2 + z_3 + \cdots \\ &= (1 + x_1 + x_1^2 + x_1^3 + \cdots)(1 + x_2 + x_2^2 + x_2^3 + \cdots)(1 \\ &+ x_3 + x_3^2 + x_3^3 + \cdots)\end{aligned}$$

That is,

$$\begin{aligned}1 + z_1 + z_2 + z_3 + \cdots \\ &= 1 + (x_1 + x_2 + x_3) \\ &+ (x_1^2 + x_1x_2 + x_1x_3 + x_2x_3 + x_2^2 + x_3^2) \\ &+ (x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2^3 \\ &+ x_3^3 + x_1x_2x_3) \\ &+ (x_1^3x_2 + x_1^3x_3 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^3 + x_1^2x_3^2 \\ &+ x_1x_2^2x_3 + x_1x_2x_3^2 + x_2^3x_3 + x_1x_3^3 + x_2^3x_3 + x_2x_3^3) \\ &+ \cdots\end{aligned}$$

By identifying the homogenous parts of (cohomological) degree r in the equation above, we obtain the following proposition:

Proposition 3.1: In cohomology dimension $r \geq 1$, we have the equality

$$z_r = \sum_{0 \leq k \leq t \leq r} {x_1}^{r-t} {x_2}^{t-k} {x_3}^{k}$$

Clearly, $z_r \ (n+1 \leq r \leq n+3)$ is the complete homogeneous symmetric polynomial of degree r in the variables x_1, x_2, x_3.

That is,

$$z_r = \sum_{0 \leq k \leq t \leq r} {x_1}^{r-t} f_t$$

Where

$$f_t = \sum_{k=0}^{t} {x_2}^{t-k} {x_3}^{k}.$$

Therefore, the ideals generated by

$$I_{1,1,1,m-3} = (z_{m+1}, z_{m+2}, z_{m+3})$$

are given as follows:

$$z_{n+1} = {x_1}^{n+1} + {x_1}^{n} f_1 + {x_1}^{n-1} f_2 + \cdots + x_1 f_n + f_{n+1}$$
$$z_{n+2} = x_1 z_{n+1} + f_{n+2}$$
$$z_{n+3} = x_1 z_{n+2} + f_{n+3}.$$

Recall: A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal I is said to be a Groebner basis if

$$\langle LT(g_1), \cdots, LT(g_t) \rangle = \langle LT(I) \rangle.$$

Corollary 3.2: Fix a monomial order. Then every ideal $I \subset k[x_1, \ldots, x_n]$ other than $\{0\}$ has a Groebner basis. Furthermore, any Groebner basis for an ideal *I* is a basis of *I*.

To produce a Groebner basis, we extend the original generating set to a Groebner basis by adding more polynomials in I. These polynomials are the remainders after dividing the S polynomials through the generating set. This remainder becomes a new generator.

Thus the ideal $I_{1,1,1,m-3} = (z_{m+1}, z_{m+2}, z_{m+3})$ is extended to Groebner basis by computing the S polynomial of each pair. Let us define the initial

$$G = \{z_{n+1}, z_{n+2}, z_{n+3}\}$$

and test if the set is a Groebner basis; if not, we get it improved till the set satisfies the required criterion.

To compute the S polynomial of (z_{n+1}, z_{n+2}), we note that the leading monial in $z_{n+1} = x_1{}^{n+1}$ and that of $z_{n+1} = x_1{}^{n+1}$ and by definition of least common monomials, $\omega = x_1{}^{n+2}$.

$$S(z_{n+1}, z_{n+2}) = \frac{x_1{}^{n+2}}{x_1{}^{n+1}}(z_{n+1}) + \frac{x_1{}^{n+2}}{x_1{}^{n+1}}(z_{n+2}) \quad = x_1(z_{n+1}) + z_{n+2}$$
$$= f_{n+2}.$$

Then, $S(z_{n+1}, z_{n+2}) \longrightarrow_G f_{n+2}$

so we add f_{n+2} to G to obtain a new set,

$$G_1 = \{z_{n+1}, z_{n+2}, z_{n+3}, f_{n+2}\}.$$

Obviously,

$$S(z_{n+1}, z_{n+2}, f_{n+2}) \longrightarrow_{G_1} 0$$

To compute the S polynomial of (z_{n+2}, z_{n+3}), we note that the leading monial in $z_{n+2} = x_1{}^{n+2}$ and that of $z_{n+3} = x_1{}^{n+3}$ and by definition of least common multiples of monomials, $\omega = x_1{}^{n+3}$

$$S(z_{n+2}, z_{n+3}) = \frac{x_1{}^{n+3}}{x_1{}^{n+2}}(z_{n+2}) + \frac{x_1{}^{n+3}}{x_1{}^{n+3}}(z_{n+3}) \quad = x_1(z_{n+2}) + z_{n+3}$$
$$= f_{n+3}$$

Thus,

$$S(z_{n+2}, z_{n+3}) \longrightarrow_{G_1} f_{n+3}$$

so we add f_{n+3} to G to get a new set,

$$G_2 = \{z_{n+1}, z_{n+2}, z_{n+3}, f_{n+2}, f_{n+3}\}.$$

Obviously, $S(z_{n+2}, z_{n+3}) \longrightarrow_{G_2} 0$

To compute the S polynomial of (z_{n+1}, z_{n+3}) , $\omega = x_1^{\,n+3}$

$$\begin{aligned} S(z_{n+1}, z_{n+3}) &= \frac{x_1^{\,n+3}}{x_1^{\,n+1}}(z_{n+1}) + \frac{x_1^{\,n+3}}{x_1^{\,n+3}}(z_{n+3}) \quad = x_1^2(z_{n+1}) + z_{n+3} \\ &= x_1 f_{n+2} + f_{n+3} \end{aligned}$$

Thus, $S(z_{n+1}, z_{n+3}) \longrightarrow_{G_2} 0$

Considering the pairs $(z_{n+1}, f_{n+2}), (z_{n+2}, f_{n+2}), (z_{n+3}, f_{n+2})$ and $(z_{n+1}, f_{n+3}), (z_{n+2}, f_{n+3}), (z_{n+3}, f_{n+3})$, the gcd of the leading terms of each pair is 1, thus by earlier proposition,

$$S(z_{n+1}, f_{n+2}) = S(z_{n+2}, f_{n+2}) = S(z_{n+3}, f_{n+2}) \longrightarrow_{G_2} 0$$
$$S(z_{n+1}, f_{n+3}) = S(z_{n+2}, f_{n+3}) = S(z_{n+3}, f_{n+3}) \longrightarrow_{G_2} 0$$

For the S polynomial of (f_{n+2}, f_{n+3}) , $\omega = x_2^{\,n+3}$,

$$\begin{aligned} S(f_{n+2}, f_{n+3}) &= \frac{x_2^{\,n+3}}{x_2^{\,n+2}}(f_{n+2}) + \frac{x_2^{\,n+3}}{x_2^{\,n+3}}(f_{n+3}) \quad = x_2(f_{n+2}) + f_{n+3} \\ &= x_3^{\,n+3}. \end{aligned}$$

Thus, $S(f_{n+2}, f_{n+3}) \longrightarrow_{G_2} x_3^{\,n+3}$

and adding $x_3^{\,n+3}$ to G_2 gives a new set,

$$G_3 = \{z_{n+1}, z_{n+2}, z_{n+3}, f_{n+2}, f_{n+3}, x_3^{\,n+3}\}$$
$$\text{with } S(f_{n+2}, f_{n+3}) \longrightarrow_{G_3} 0.$$

Thus the algorithm terminates and we have

$$G_3 = \{z_{n+1}, z_{n+2}, z_{n+3}, f_{n+2}, f_{n+3}, x_3^{\,n+3}\}$$

as the Groebner basis.

However, from results on Groebner basis (see Becker and Weispfenning, 1993) and definition of reduced Groebner basis, some of these bases are redundant and removing them will not affect the set. The leading terms of z_{n+2} and z_{n+3} are multiples of the leading term of z_{n+1}, so we can drop z_{n+2} and z_{n+3} from the list. Similarly the leading term of f_{n+3} is a multiple of the leading term of f_{n+2}, so f_{n+3} is removed from the list. This process leads to the set

$$G = \{z_{n+1}, f_{n+2}, x_3^{n+3}\}$$

This implies that the ideal generated by $z_{n+1}, f_{n+2}, x_3^{n+3}$ coincides with the ideal

$$I_{1,1,1,m-3} = (z_{m+1}, z_{m+2}, z_{m+3})$$

that is the set

$$\{z_{n+1}, f_{n+2}, x_3^{n+3}\}$$

is a basis for $I_{1,1,1,n-3}$.

Let $\prec$ be the lexicographic term ordering (lex ordering) in $Z_2[x_1, x_2, x_3]$ with $x_1 > x_2 > x_3$ such that $x_1{}^{\alpha}x_2{}^{\beta}x_3{}^{\gamma} \prec x_1{}^{a}x_2{}^{b}x_3{}^{c}$ if and only if $\alpha < a$ or else $\alpha = a$ and $\beta \leq b$.

Theorem 3.3: The The set $\{z_{n+1}, f_{n+2}, x_3^{n+3}\}$ is the reduced Groebner basis $I_{1,1,1,n-3}$ with respect to the lexicographic ordering $\prec$.

Proof: It is clear that the leading terms of $z_{n+1}, f_{n+2}, x_3^{n+3}$ are LT(z_{n+1}) = x_1^{n+1} , LT(f_{n+2}) = x_2^{n+2} LT(x_3^{n+3}) $= x_3^{n+3}$,respectively. Thus, $gcd(x_1^{n+1}, x_2^{n+2}, x_3^{n+3}) = 1$. So the conditions are satisfied.

Next, we show that $g_i \subseteq I_{1,1,1,n-3}$ $0 \leq i \leq n+2$, where $G = \{g_0, g_1, \cdots, g_{n+2}\}$. It is obvious that $z_{n+1} \in I_{1,1,1,n-3}$, and we now show that $f_{n+2}, x_3^{n+3} \in I_{1,1,1,n-3}$.

Recall that from equation above,

$$x_1(z_{n+1}) + z_{n+2} = f_{n+2}.$$

This implies that

$$f_{n+2} \in < z_{n+1}, z_{n+2} > \in I_{1,1,1,n-3}$$

Also,

$$\begin{aligned} x_3^{n+3} &= x_1(q_{n+2} + x_2(f_{n+2})) = x_1(q_{n+2}) + x_2(x_1(z_{n+1}) + z_{n+2}) \\ &= (x_1 + x_2)z_{n+2} + x_1 x_2(z_{n+1})) \end{aligned}$$

This implies that

$$x_3^{n+3} \in < z_{n+1}, z_{n+2} > \in I_{1,1,1,n-3}.$$

Therefore,

$$< z_{n+1}, z_{n+2}, z_{n+3} > = < z_{n+1}, f_{n+2}, x_3^{n+3} >.$$

Once a Groebner basis G for an ideal in the polynomial algebra is obtained, an additive basis for the quotient algebra could be formed by taking all terms (more precisely, their classes) which are not divisible by any of the leading terms in G (see Buchberger, (1976) and Cox, Little, O'Shea (2012)). Since the leading terms LT$(z_{n+1}) = x_1^{n+1}$, LT$(f_{n+2}) = x_2^{n+2}$ LT$(x_3^{n+3}) = x_3^{n+3}$ and the term $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$ is not divisible by any of the leading terms LT(g_i) if and only if $\alpha_i \le n + i - 1$ for all $i = \{1, 2, 3\}$.

Corollary 3.4: If $x_1, x_2, x_3 \in H^1(F(1,1,1, m-3))$ are Stiefel-Whitney classes of three canonical line bundles $\gamma_1, \gamma_2, \gamma_3$ over $F(1,1,1, m-3)$, then the set $\{x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} \mid \alpha_i \le m + i - 1\}$ is a vector space basis for $H^*(F(1,1,1, m-3), Z_2)$.

An example is given to illustrate the use of Groebner bases.

Example 3.5: The height of a cohomology class σ, denoted by $ht(\sigma)$, is the maximum m such that $\sigma^m \neq 0$. By the previous corollary, $x_i^{m+i-1} \neq 0$ in $H^*(F(1,1,1, m-3), Z_2)$.

Proposition 3.6: Every cohomology class $\sigma \in H^*(F(1,1,1, m-3), Z_2)$ can be written in the form $\sigma = \sum_{i=0}^{n} x_1^{i} p_i(x_2, x_3)$ where $p_i(x_2, x_3)$ are polynomials in variables x_2, x_3 only. Moreover, $\sigma = 0$ if and only if $p_i(x_2, x_3)=0$ in $Z_2[x_1, x_2, x_3]/(G) \cong H^*(F(1,1,1, m-3), Z_2)$, for all $i = \{0, 1, \cdots, n\}$

Now $g_0 = z_{m+1}$ is an element of the ideal $I_{1,1,1,m-3}$ so, $z_{m+1} = 0$ in $H^*(F(1,1,1, m-3), Z_2) \cong Z_2[x_1, x_2, x_3]/I_{1,1,1,m-3}$. By the earlier Proposition,

$$z_{n+1} = x_1^{n+1} + x_1^{n} f_1 + x_1^{n-1} f_2 + \cdots + x_1 f_n + f_{n+1} = 0$$
$$\Longrightarrow x_1^{n+1} = x_1^{n} f_1 + x_1^{n-1} f_2 + \cdots + x_1 f_n + f_{n+1}$$

Since the leading term LT$(z_{n+1}) = x_1^{n+1}$, is a sum of finite basis elements and by the last Proposition, we conclude that $x_1^{n+1} \neq 0$. Hence, the height, $ht(x_1) = n + 1$.

For f_{n+2} , whose leading term is x_2^{n+2} , in a similar way, and by the last Proposition, the height, $ht(x_2) = n + 1$ since f_{n+2} is also a sum of finite basis elements, while the $ht(x_3) = 0$ since $x_3^{n+3} \in I_{1,1,1,m-3}$.

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